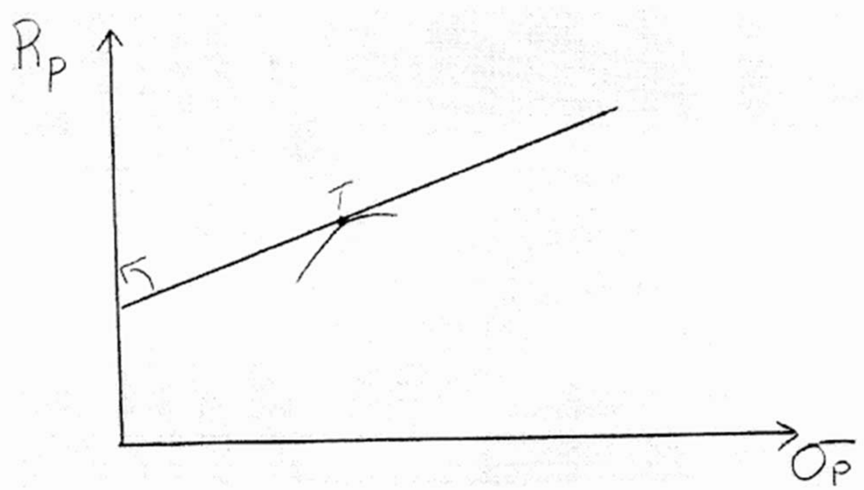
SOLVING FOR THE EFFICIENT FRONTIER

Elton & Gruber

Chapter 6



Recall T is point on the efficient frontier in most counter-clockwise directions. Equation of line Tangent at T is:

$$\bar{R}_i = R_F + \frac{\bar{R}_T - R_F}{\sigma_T}\sigma_i$$

Moving in most counter-clockwise directions is equivalent to maximizing slope. Thus, the objective is to find portfolio where:

$$\theta = \frac{\bar{\mathrm{R}}_{\mathrm{T}} - \mathrm{R}_{\mathrm{F}}}{\sigma_{\mathrm{T}}}$$

is maximum.

$$\Theta = \frac{\Sigma X_i \bar{R}_i - 1R_F}{\left[\Sigma X_i^2 \sigma_i^2 + \Sigma\Sigma X_i X_j \sigma_{ij}\right]^{1/2}}$$

Subject to $\Sigma X_i = 1$

$$\Theta = \frac{\Sigma X_i \bar{R}_i - \left[\Sigma X_i\right] R_F}{\left[\Sigma X_i^2 \sigma_i^2 + \Sigma\Sigma X_i X_j \sigma_{ij}\right]^{1/2}}$$

$$= \frac{\Sigma X_i \left(\bar{R}_i - R_F\right)}{\left[\Sigma X_i^2 \sigma_i^2 + \Sigma\Sigma X_i X_j \sigma_{ij}\right]^{1/2}}$$

Θ can be written as:

$$\Theta = \left(\bar{R}_T - R_F\right)\left(\sigma^2\right)^{-1/2}$$

recall how to take a derivative using the chain rule namely:

$$d\Theta = f(X,Y) = XdY + YdX$$

and that

$$dX^N = NX^{N-1}dx$$

thus

$$\frac{d\Theta}{dX_k} = \left(\bar{R}_T - R_F\right)\left(-\tfrac{1}{2}\right)\left(\sigma^2\right)^{-3/2}\frac{d\sigma^2}{dX_k} + \frac{d\left[\bar{R}_T - R_F\right]\left(\sigma^2\right)^{-1/2}}{dX_k}$$

or

$$\frac{d\Theta}{dX_k} = \left(\bar{R}_T - R_F\right)\left(-\tfrac{1}{2}\right)\left(\sigma^2\right)^{-3/2}\left[2X_k\sigma_k^2 + 2\sum_{k\neq i} X_i\sigma_{ik}\right] + \left(\bar{R}_k - R_F\right)\left(\sigma^2\right)^{-1/2}$$

$$\frac{d\Theta}{dX_k} = -\frac{\bar{R}_T - R_F}{\sigma^2}\left[X_k \sigma_k^2 + \sum_{k \neq i} X_i \sigma_{ik}\right] + \left(\bar{R}_k - R_F\right)$$

$$\left(\bar{R}_k - R_F\right) = \lambda\left[X_k \sigma_k^2 + \sum_{k \neq i} X_i \sigma_{ik}\right]$$

where:

$$\lambda = \frac{\bar{R}_T - R_F}{\sigma^2}$$

First Order Condition

$$\bar{R}_k - R_F = \lambda\left[X_1\sigma_{1k} + X_2\sigma_{2k} + \ldots X_k\sigma_k^2 + \ldots + X_N\sigma_{Nk}\right]$$

for all i

where $\lambda = \dfrac{\bar{R}_T - R_F}{\sigma_T^2}$

define $Z_i = \lambda X_i$ then

$$\bar{R}_k - R_F = Z_1\sigma_{ik} + Z_2\sigma_{2k} + \ldots Z_k\sigma_k^2 + \ldots + Z_N\sigma_{Nk}$$ all k

Example 3 securities

$$\bar{R}_1 - R_F = Z_1\sigma_1^2 + Z_2\sigma_{21} + Z_3\sigma_{31}$$

$$\bar{R}_2 - R_F = Z_1\sigma_{21} + Z_2\sigma_2^2 + Z_3\sigma_{32}$$

$$\bar{R}_3 - R_F = Z_1\sigma_{31} + Z_2\sigma_{23} + Z_3\sigma_3^2$$

X's are $X_i = \frac{Z_i}{\Sigma_j Z_j}$

$$\bar{R} = \begin{pmatrix} \bar{R}_1 - R_F \\ \bar{R}_2 - R_F \\ \bar{R}_3 - R_F \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sigma_1^2 \sigma_{12} \sigma_{13} \\ \sigma_{12} \sigma_2^2 \sigma_{23} \\ \sigma_{13} \sigma_{23} \sigma_3^2 \end{pmatrix}$$

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}$$

$$\bar{R}' = \sum Z$$

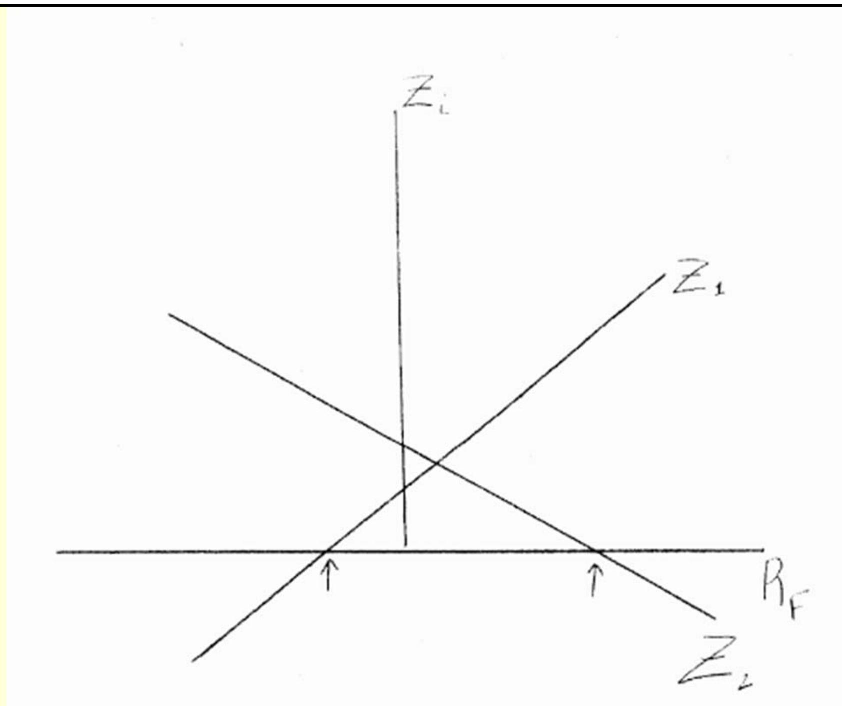
and

$$Z = \Sigma^{-1} \bar{R}'$$

Typical Element:

$$Z_i = C_0 + C_1 R_F$$

Thus



No Riskless Borrowing or Lending, but Short Sales are Allowed

(Here, there are an infinite number of efficient risky-asset only portfolios)

1. Choose a riskfree rate, say, 1%
2. Solve for the efficient portfolio
3. Choose another riskfree rate, say 2%
4. Solve for the efficient portfolio
5. Repeat above steps until entire efficient portfolio has been traced out

<u>No</u> Riskless Borrowing or Lending, and **<u>no</u>** Short Sales

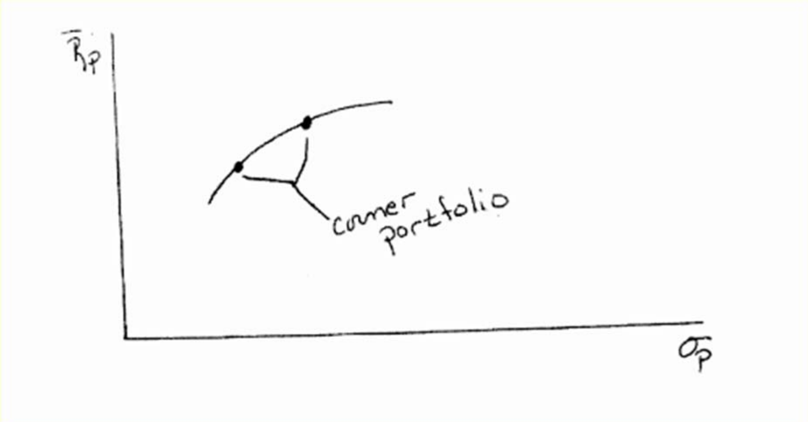
Minimize σ^2

Subject to:

1. $\Sigma X_i = 1$
2. $\Sigma X_i \bar{R}_i = \bar{R}_p$
3. $\Sigma X_i \geq 0$

Quadratic Programming Problem

Since have squared and cross product terms.



Between corner portfolios, a portfolio consists of linear combination of corner. At corner securities enter or leave portfolio.

Riskless Borrowing or Lending, and **no** Short Sales

(Same as base case, but with one extra constraint):

$$\Sigma X_i \geq 0$$

This, again, involves quadratic programming optimization methods to solve—since the inequalities cannot be substituted into the objective function anymore!